

## Summary of Tests for Convergence of Series

<b>Geometric Series</b>	The geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with ratio $r$ converges for $-1 < r < 1$ and diverges for $ r  \geq 1$ .	The sum of the convergent geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ is $\frac{a}{1-r}$ . The $N$ -th partial sum is given by $S_N = \sum_{k=1}^N ar^{k-1} = a \cdot \frac{1-r^N}{1-r}$
<b>p-Series</b>	The $p$ -series $\sum \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$ . This includes the special case of the <i>Harmonic series</i> for $p = 1$ , which diverges.	
<b>Divergence Test</b>	If the sequence $a_n$ does not converge to 0, then the series $\sum a_k$ diverges.	This is the first test to apply because the conclusion is simple. However, if $\lim_{n \rightarrow \infty} a_n = 0$ , no conclusion can be drawn.
<b>Integral Test</b>	Let $f$ be a positive, decreasing function on an interval $[c, \infty]$ and let $a_k = f(k)$ for each positive integer $k \geq c$ . <ul style="list-style-type: none"> <li>If <math>\int_c^{\infty} f(t) dt</math> converges, then <math>\sum a_k</math> converges.</li> <li>If <math>\int_c^{\infty} f(t) dt</math> diverges, then <math>\sum a_k</math> diverges.</li> </ul>	Use this test when $f(x)$ is easy to integrate.
<b>Direct Comparison Test</b>	Let $0 \leq a_k \leq b_k$ for each positive integer $k$ . <ul style="list-style-type: none"> <li>If <math>\sum b_k</math> converges, then <math>\sum a_k</math> converges.</li> <li>If <math>\sum a_k</math> diverges, then <math>\sum b_k</math> diverges.</li> </ul>	Use this test when you have a series with known behavior that you can compare to – this test can be difficult to apply.
<b>Limit Comparison Test</b>	Let $a_n$ and $b_n$ be sequences of positive terms. If $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L$ for some positive finite number $L$ , then the two series $\sum a_k$ and $\sum b_k$ either both converge or both diverge.	Easier to apply in general than the comparison test, but you must have a series with known behavior to compare. Useful to apply to series of rational functions.
<b>Ratio Test</b>	Let $a_k \neq 0$ for each $k$ and suppose $\lim_{k \rightarrow \infty} \frac{ a_{k+1} }{ a_k } = r.$ <ul style="list-style-type: none"> <li>If <math>r &lt; 1</math>, then the series <math>\sum a_k</math> converges absolutely.</li> <li>If <math>r &gt; 1</math>, then the series <math>\sum a_k</math> diverges.</li> <li>If <math>r = 1</math>, then test is inconclusive.</li> </ul>	This test is useful when a series involves factorials and powers.
<b>Root Test</b>	Let $a_k \geq 0$ for each $k$ and suppose $\lim_{k \rightarrow \infty} \sqrt[k]{a_k} = r.$ <ul style="list-style-type: none"> <li>If <math>r &lt; 1</math>, then the series <math>\sum a_k</math> converges.</li> <li>If <math>r &gt; 1</math>, then the series <math>\sum a_k</math> diverges.</li> <li>If <math>r = 1</math>, then test is inconclusive.</li> </ul>	In general, the Ratio Test can usually be used in place of the Root Test. However, the Root Test can be quick to use when $a_k$ involves $k$ th powers.
<b>Alternating Series Test</b>	If $a_n$ is a positive, decreasing sequence so that $\lim_{n \rightarrow \infty} a_n = 0$ , then the alternating series $\sum (-1)^{k+1} a_k$ converges.	This test applies only to alternating series – we assume that the terms $a_n$ are all positive and that the sequence $\{a_n\}$ is decreasing.
<b>Alternating Series Estimation Theorem</b>	Let $S_n = \sum_{k=1}^n (-1)^{k+1} a_k$ be the $n$ th partial sum of the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ . Assume $a_n > 0$ for each positive integer $n$ , the sequence $a_n$ decreases to 0 and $\lim_{n \rightarrow \infty} S_n = S$ . Then it follows that $ S - S_n  < a_{n+1}$ .	This bound can be used to determine the accuracy of the partial sum $S_n$ as an approximation of the sum of a convergent alternating series.

# SERIES CONVERGENCE/DIVERGENCE FLOW CHART

## TEST FOR DIVERGENCE

Does  $\lim_{n \rightarrow \infty} a_n = 0$ ?

NO

$\sum a_n$  Diverges

YES

## p-SERIES

Does  $a_n = 1/n^p, n \geq 1$ ?

YES

Is  $p > 1$ ?

YES

$\sum a_n$  Converges

NO

$\sum a_n$  Diverges

NO

## GEOMETRIC SERIES

Does  $a_n = ar^{n-1}, n \geq 1$ ?

YES

Is  $|r| < 1$ ?

YES

$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$

NO

$\sum a_n$  Diverges

NO

## ALTERNATING SERIES

Does  $a_n = (-1)^n b_n$  or  $a_n = (-1)^{n-1} b_n, b_n \geq 0$ ?

YES

Is  $b_{n+1} \leq b_n$  &  $\lim_{n \rightarrow \infty} b_n = 0$ ?

YES

$\sum a_n$  Converges

NO

## TELESCOPING SERIES

Do subsequent terms cancel out previous terms in the sum? May have to use partial fractions, properties of logarithms, etc. to put into appropriate form.

YES

Does  $\lim_{n \rightarrow \infty} s_n = s$  finite?

YES

$\sum a_n = s$

NO

$\sum a_n$  Diverges

NO

## TAYLOR SERIES

Does  $a_n = \frac{f^{(n)}(a)}{n!} (x-a)^n$ ?

YES

Is  $x$  in interval of convergence?

YES

$\sum_{n=0}^{\infty} a_n = f(x)$

NO

$\sum a_n$  Diverges

NO

Try one or more of the following tests:

## COMPARISON TEST

Pick  $\{b_n\}$ . Does  $\sum b_n$  converge?

YES

Is  $0 \leq a_n \leq b_n$ ?

YES

$\sum a_n$  Converges

NO

Is  $0 \leq b_n \leq a_n$ ?

YES

$\sum a_n$  Diverges

NO

## LIMIT COMPARISON TEST

Pick  $\{b_n\}$ . Does  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$  finite &  $a_n, b_n > 0$ ?

YES

Does  $\sum_{n=1}^{\infty} b_n$  converge?

YES

$\sum a_n$  Converges

NO

$\sum a_n$  Diverges

## INTEGRAL TEST

Does  $a_n = f(n), f(x)$  is continuous, positive & decreasing on  $[a, \infty)$ ?

YES

Does  $\int_a^{\infty} f(x) dx$  converge?

YES

$\sum_{n=a}^{\infty} a_n$  Converges

NO

$\sum a_n$  Diverges

## RATIO TEST

Is  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| \neq 1$ ?

YES

Is  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ ?

YES

$\sum a_n$  Abs. Conv.

NO

$\sum a_n$  Diverges

## ROOT TEST

Is  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \neq 1$ ?

YES

Is  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ ?

YES

$\sum a_n$  Abs. Conv.

NO

$\sum a_n$  Diverges