The Intermediate Value Theorem

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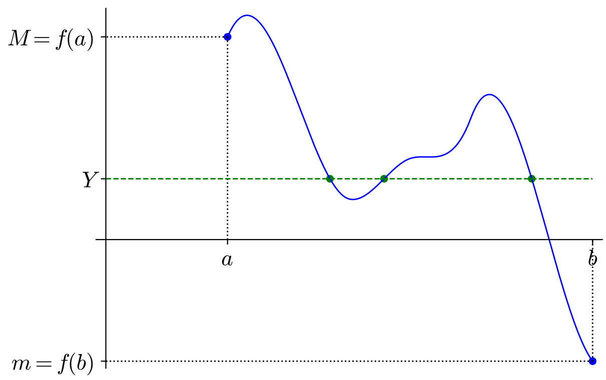
Often we are interested in studying the sign of a function on its domain, for example:

- The **sign of a function** on an interval tells us whether the function is positive or negative on that interval, and this tells use whether to graph the function above or below the x-axis on that interval.
- The **sign of the derivative** of the function on an interval tells us if the original function is increasing or decreasing on that interval.
- The **sign of the second derivative** of the function tells us if the original function is concave up or concave down on that interval.

The interpretation depends whether we are studying the sign of f, f', or f'', but the technique used is the same and consists of finding a sign chart for the function under consideration. But to be able to do this, we actually need an important result from mathematical analysis for continuous functions:

> **Theorem (Intermediate Value Theorem):** If f is continuous on the closed interval $[a, b]$, then f must take all possible y-values between $f(a)$ and $f(b)$ at least once somewhere on this interval. More precisely, if $m = \min(f(a), f(b))$ and $M = \max(f(a), f(b))$, and $Y \in [m, M]$, then there exists $c \in [a, b]$ such that $f(c) = Y$.

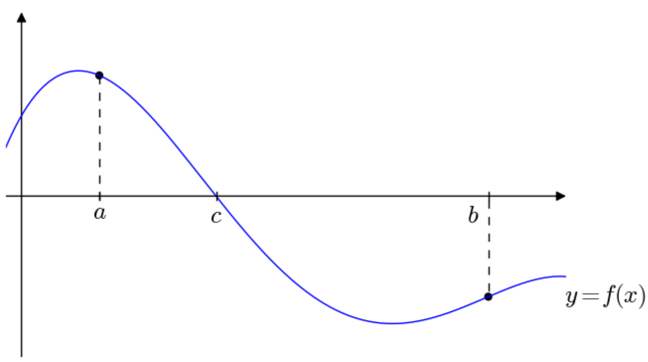
The intuition behind this statement is shown in the figure on the right: Since the graph of a continuous function can be drawn in one uninterrupted pen-stroke, the graph cannot "avoid" any "intermediate" y-values when trying to connect $(a, f(a))$ to $(b, f(b))$. Note that this only tells us about the existence of an x-value c where the y-value is attained, but it doesn't tell us where exactly that is — and it is in fact possible that there is more than one possibility (as shown in the figure). While intuitively clear, proving this statement rigorously takes some work, but we will omit this here.



An important special case of this theorem is when the y-value of interest is 0:

> **Theorem (Intermediate Value Theorem — Root Variant):** If f is continuous on the closed interval $[a, b]$ and $f(a)f(b) < 0$ (that is $f(a)$ and $f(b)$ have different signs), then there exists $c \in (a, b)$ such that c is a root of f, that is $f(c) = 0$.

Note that this only guarantees the *existence* of a root, but it does not tell us where exactly this root occurs.
In this form, the Intermediate Value Theorem can be used to prove that an equation must have *at least* one solution — and this can be used to design numerical algorithms to estimate the roots of a function (the "bisection method" for example).



Example: Show that the equation $\cos x = x$ has a solution.

Solution: Write the equation as $\cos x - x = 0$ and notice that the equation has a solution if the function $f(x) = \cos x - x$ has a root. This function is the difference of two continuous functions on $\mathbb{R}$ and therefore is also continuous on $\mathbb{R}$. Since the problem is equivalent to a root finding problem, the Intermediate Value Theorem is the right tool to use. We can show that there is a root, by providing an interval $[a, b]$ where the continuous function f changes sign at the endpoints of the interval. Now $f(0) = \cos 0 - 0 = 1$ is positive, so if we can find a point where f is negative we are done. Let's try $x = \pi/2$: we have $f(\pi/2) = \cos \pi/2 - \pi/2 = 0 - \pi/2 < 0$. Therefore, there exists a point $c \in (0, \pi/2)$ where $f(c) = 0$, or equivalently, where $\cos c = c$. What the theorem does not tell us is whether this is the only solution of the equation.

$\diamondsuit$

Thinking about the converse of the Intermediate Value Theorem gives an important consequence:

Corollary: If f is continuous on the interval $[a, b]$ and does *not* have a root in this interval (i.e. $f(x) \neq 0$ for all $x \in [a, b]$), then f cannot change signs in this interval. In other words, either $f(x) > 0$ for all $x \in [a, b]$, or $f(x) < 0$ for all $x \in [a, b]$.

Proof: We prove this statement by contradiction: Assume that f has no roots in $[a, b]$, but that it does change sign somewhere within the interval. Then there would be some values c and d in this interval, with $a < c < d < b$, where the sign of f at $x = c$ would be the opposite of the sign of f at $x = d$. But by the Intermediate Value Theorem this would mean that somewhere between c and d (and therefore somewhere in the interval (a, b)), there would have to be a root of f, which is a contradiction to our assumption. $\blacksquare$

In summary, this immediately implies:

Corollary: A function can only change sign where it has a root or where it is not continuous.

As an example, consider the function $f(x) = 1/x$: it is discontinuous at $x = 0$ and changes sign at this point.

All these results together give us a systematic method to study the sign of a function (which we have already used):

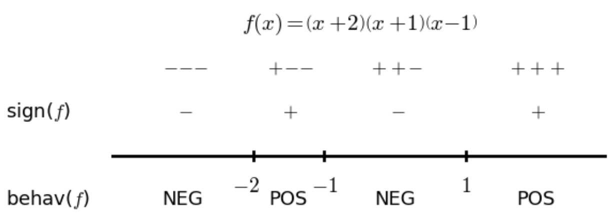
1. Find all roots of the function and all x-values where the function is not defined. These x-values are *candidates* for the location of a possible sign change.
2. In each interval inbetween, decide the sign (for example by using a test value).
3. Summarize the information in a sign chart.

Example: Consider the function $f(x) = x^3 + 2x^2 - x - 2$. Study the sign of the functions f, f', and f''. Give an interpretation in terms of the original function f in each case.

Solution: We must first find the roots of f by factoring the polynomial. We have

$$f(x) = x^3 + 2x^2 - x - 2 = x^2(x + 2) - (x + 2) = (x + 2)(x^2 - 1) = (x + 2)(x - 1)(x + 1).$$

Therefore the roots of f are -2, -1, and 1. Since f is a polynomial, it is continuous everywhere and the function can only change sign between its roots. When $x < -2$, the factors in the expression $(x + 2)(x - 1)(x + 1)$ are negative, negative, and negative. We may summarize this as $- - -$ and overall, the product of three negative values is negative. When $-2 < x < -1$, the factors are positive, negative, and negative: $+ - -$. Overall, the function is positive on that interval. For $-1 < x < 1$, the factors are $+ - +$, and f is negative. For $x > 1$, the factors are $+ + +$ and f is positive. We can summarize this information in a sign chart which is shown in Figure 2.



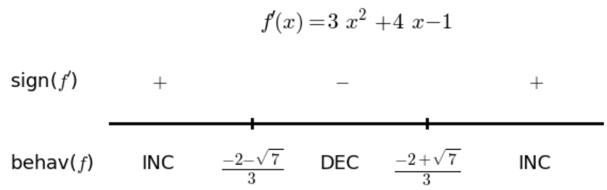
The first derivative of f is $f'(x) = 3x^2 + 4x - 1$.

Factoring this polynomial is possible (although not with integers), but all we really need are the roots, which can be obtained using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. We obtain

$$x = \frac{-4 \pm \sqrt{28}}{6} = \frac{-4 \pm 2\sqrt{7}}{6} = \frac{-2 \pm \sqrt{7}}{3}.$$

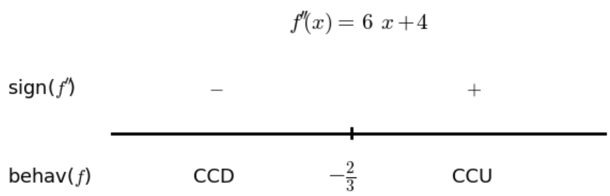
Without the factorization of the polynomial it is also possible to study the sign of a function by substituting test values and noting the sign of the function. For instance, when $x < (-2 - \sqrt{7})/3$, we can use $x = -2$ as a test value, since this is less than $(-2 - \sqrt{7})/3$. Since $f'(-2) = 12 - 8 - 1 = 3 > 0$, f' is positive on that interval.

When $(-2 - \sqrt{7})/3 < x < (-2 + \sqrt{7})/3$, we can use $x = 0$ as a test value. Since $f'(0) = -1 < 0$, f' is negative on that interval. When $x > (-2 + \sqrt{7})/3$, we can use $x = 1$ and since $f'(1) = 3 + 4 - 1 = 6 > 0$, f' is positive on that interval.[1] This gives the following sign chart for f':

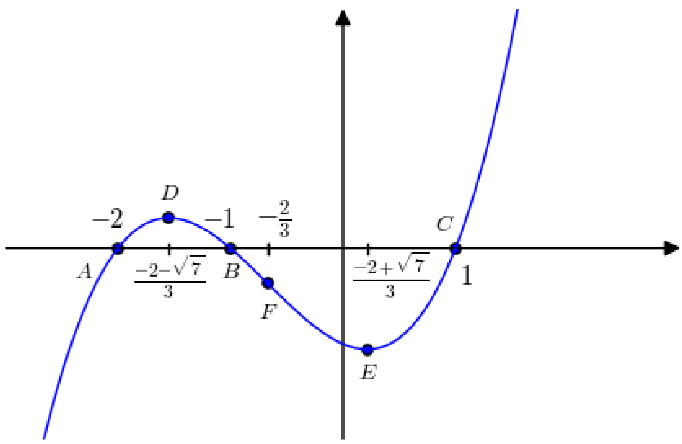


The second derivative f'' is $f''(x) = 6x + 4 = 6(x + 2/3)$.

The only root of this function is $x = -2/3$. The function is positive if $x > -2/3$ and negative if $x < -2/3$, resulting in the following sign chart for f'':



In this example, putting all the information gathered by studying the sign of f, f', and f'' allows us to sketch the graph of f and identify all relevant points (maxima, minima, inflection points, and x-intercepts):



[1]Since f' is a quadratic polynomial, we also know that its graph is a parabola. Because the leading coefficient is 3, this parabola is concave up and therefore lies below the x-axis between its roots. Therefore f' is only negative between its two roots. This is another way to study the sign of f', but it only works here because f' is a simple function. Otherwise, studying the sign of the function between its roots and points of discontinuity is a method that always works. Of course, this means that we can find the roots of the function easily (which cannot always be done) and that usually implies that we can factor the function easily as well.